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On the thermoelastic stresses of multiple interacting inhomogeneities

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Abstract

An analytic solution is presented for the two-dimensional thermoelastic problem of multiple interacting circular inhomogeneities of different sizes and thermoelastic properties embedded in an isotropic elastic medium. Based upon the complex potentials of Muskhelishvili, the analytic solution is derived for the single circular inhomogeneity problem under arbitrary thermal loadings. The solution is then applied to the problem of an infinitely extended medium containing randomly located multiple inhomogeneities successively. This procedure leads to a series solution derived with perturbation technique. Study examples show the elegance and robustness of the present approach. The results reveal the dependence of the resulting thermal stresses upon the mismatch of the thermoelastic properties and the configuration of the inhomogeneities. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The advanced engineering composite materials are benefited from taking of the dissimilar properties of different materials with superior properties. However, the large mismatch in thermoelastic properties of the different material phases in advanced composites produces severe thermal stresses under thermal loading. For example, the high residual thermal stresses developed in a metal matrix composite during cooling from consolidation temperatures may be large enough to initiate microcracks in the matrix phase adjacent to the fiber/matrix interface or plastic yielding before application of external forces. Along or combined with stresses caused by external forces, the thermal stresses can be the cause of brittle fracture and fatigue failure (Coffin, 1954; Manson, 1954), plastic damages (Parkes, 1954),

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deterioration in material properties (Fazekas, 1953). Such problems have focused interest on the thermal stresses induced by the mismatch of different materials in composites, which can be idealized as a two-dimensional thermoelastic problem of circular inhomogeneities embedded in an isotropic medium.

The problem has been extensively studied for single inhomogeneity under particular loading conditions, e.g., the works of Mindlin and Cooper (1950), Florence and Goodier (1959, 1960, 1963), Tauchert (1968), Podil'chuk and Kirilyuk (1988), Edmonds and Tweed (1988), Lee and Choi (1989), Theocaris and Bardzokas (1989), Zashkil'nyak (1990), Goshima and Miyao (1990), Lee (1991), Hasebe et al. (1992), Chandrasekharaiah and Murthy (1993), and Kattis and Meguid (1995). Recently further attentions have been paid to the composite materials containing two interacting inhomogeneities or an infinite number of periodically distributed inhomogeneities, e.g., Yamada (1989, 1990), Kouris and Tsuchida (1991), Muller and Schmauder (1993). However, due to the inherent difficulty involved, few attempts have been made to solve the problem of a medium containing randomly spaced multiple interacting inhomogeneities of arbitrary sizes and different thermoelastic properties subjected to an arbitrary thermal loading.

The purpose of the present study is to provide an analytic solution to the two-dimensional thermoelastic problem of multiple interacting circular inhomogeneities. The analysis is based upon the complex stress and temperature potentials of Muskhelishvili (1955) and the Laurent expansion method in solving the problem of a single inhomogeneity. The solution for a single inhomogeneity is then applied successively to the problem of multiple inhomogeneities by an appropriate superposition. The resulting problem is solved by perturbation technique. An excellent agreement between the present and previous results for one and two inhomogeneities is observed. The interaction of multiple inhomogeneities (up to 5) has been studied and the dependence of thermal stresses upon the mismatch of thermoelastic properties of inhomogeneity and matrix phases and the configuration of inhomogeneity is discussed.

This article has been divided into five sections. Following this brief introduction, Section 2 gives the governing equations and the analytic solution for the single circular inhomogeneity problem. Section 3 describes the successive series approach for the solution of the multiple interacting circular inhomogeneity problem. These inhomogeneities may have different thermoelastic properties and sizes. The configuration of the inhomogeneities is arbitrary provided that they do not overlay. Section 4 is devoted to the application of the analytic solution obtained. A number of cases were discussed. Section 5 concludes the article.

2. Theoretical development

2.1. The governing equations

In plane thermoelastic problems, the temperature change T, the heat flux (q_x, q_y), the displacements (u, v) and the stresses $(\sigma_x, \sigma_y, \tau_{xy})$ of the uncoupled stationary thermoelastic problem can be expressed in terms of Muskhelishvili (1953) complex potentials $W(z)$, $\phi(z)$ and $\psi(z)$ as

$$T = W(z) + \overline{W(z)} \tag{1}$$

$$q_x = -k\left[W'(z) + \overline{W'(z)}\right] \tag{2}$$

$$q_y = -ik\left[W'(z) - \overline{W'(z)}\right] \tag{3}$$

$$Q = \int q_n \, \mathrm{d}s = ik[W'(z) - \overline{W'(z)}] \tag{4}$$

$$2\mu(u + iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} + A\int W(z)\,\mathrm{d}z \tag{5}$$

$$\sigma_x + \sigma_y = 2[\phi'(z) + \overline{\phi'(z)}] \tag{6}$$

$$\sigma_y - \sigma_x + i2\tau_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)] \tag{7}$$

with

Plane strain: $\kappa = 3 - 4v$ $\quad A = 4\mu a(1 + v)$

Plane stress: $\kappa = (3 - v)/(1 + v)$ $\quad A = 4\mu a$

where Q is the total heat flowing into an area and μ, v, k, a are the shear modulus, Poisson's ratio, thermal conductivity and coefficient of linear thermal expansion, respectively. The overbar in eqns (1)–(7) represents the complex conjugate and the prime denotes differentiation with respect to the argument.

The components of stresses can also be expressed in polar coordinates as:

$$\sigma_\theta + \sigma_r = 2[\phi'(z) + \overline{\phi'(z)}] \tag{8}$$

$$\sigma_r - i\tau_{r\theta} = \phi'(z) + \overline{\phi'(z)} - z\phi''(z) - \frac{z}{z}\psi'(z) \tag{9}$$

where $z = re^{i\theta}$. The corresponding Airy's stress function U is determined by $\phi(z)$ and $\psi(z)$

$$U = \mathrm{Re}\,[\bar{z}\phi(z) + \chi(z)] \quad \text{with} \quad \chi'(z) = \psi(z) \tag{10}$$

Consider an infinite isotropic elastic matrix containing a number of circular inhomogeneities, which may be finite or infinite, subjected to an arbitrary stationary thermal loading at infinity as shown in Fig. 1. It is assumed that the inhomogeneities are perfectly bonded to the matrix and no heat source exists in the matrix and the inhomogeneity. Without loss of generality, confine our attention to the j-th circular inhomogeneity with radius R_j centered at the origin O_j of the local coordinate systems (x_j, y_j) and (r_j, θ_j). All the quantities associated with the j-th inhomogeneity are distinguished by the subscript j.

The assumption of perfect bondness between the inhomogeneity and the matrix leads to the continuity of the temperature T, the total heat flow Q, the displacements (u, v) and the stresses $(\sigma_r, \tau_{r\theta})$ across the interface between the matrix and the inhomogeneity, such that,

$$T_\mathrm{M} = T_\mathrm{I} \tag{11}$$

$$Q_\mathrm{M} = Q_\mathrm{I} \tag{12}$$

$$(u + iv)_\mathrm{M} = (u + iv)_\mathrm{I} \tag{13}$$

$$(\sigma_r - i\tau_{r\theta})_\mathrm{M} = (\sigma_r - i\tau_{r\theta})_\mathrm{I} \tag{14}$$

where the subscripts 'M' and 'I' represent the matrix and the inhomogeneity, respectively.

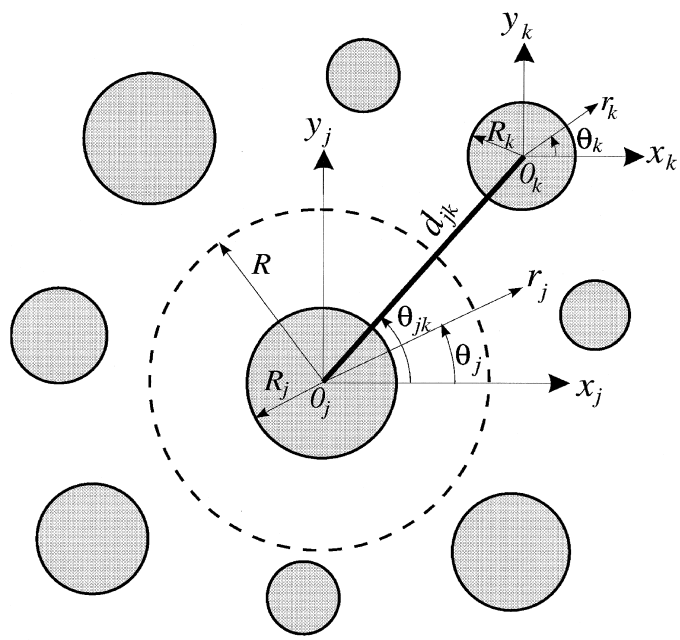

Fig. 1. A set of inhomogeneities in an isotropic elastic matrix.

Substituting eqns (1), (4), (5), (9) into (11)–(14), the continuity conditions at the interface between the matrix and the inhomogeneity can be expressed in terms of the complex potentials, such as,

$$W(t_j) + \overline{W(t_j)} = W_j(t_j) + \overline{W_j(t_j)} \tag{15}$$

$$W(t_j) - \overline{W(t_j)} = \lambda_j[W_j(t_j) - \overline{W_j(t_j)}] \tag{16}$$

$$\Gamma_j(\kappa\phi(t_j) - t_j\overline{\phi'(t_j)} - \overline{\psi(t_j)} + A\int W(t_j)\,\mathrm{d}z) = \kappa_j\phi_j(t_j) - t_j\overline{\phi_j'(t_j)} - \overline{\psi_j\ (t_j)} + A_j\int W_j(t_j)\,\mathrm{d}z \tag{17}$$

$$\phi'(t_j) + \overline{\phi'(t_j)} - t_j\phi''(t_j) - \frac{t_j}{\bar{t}_j}\psi'(t_j) = \phi_j'(t_j) + \overline{\phi_j'(t_j)} - t_j\phi_j''(t_j) - \frac{t_j}{\bar{t}_j}\psi_j'(t_j) \tag{18}$$

where the potentials with subscript j are defined inside the inhomogeneity. The variable $t_j = R_j e^{i\theta}$ is a point along the interface and the symbols $\lambda_j = k_j/k$ and $\Gamma_j = \mu_j/\mu$ are the normalized heat conductivity and shear modulus of the j-th inhomogeneirty, respectively.

2.2. Heat conduction solution

The heat conduction continuity conditions (15) and (16) can be further simplified into one equation by adding the two equations

$$W(t_j) = \frac{1+\lambda_j}{2}W_j(t_j) + \frac{1-\lambda_j}{2}\overline{W_j(t_j)} \tag{19}$$

Let $W_0(z_j)$ be a holomorphic temperature potential corresponding to the temperature state at infinity, the general solution of the heat conduction problem of a single inhomogeneity can be easily derived from eqn (19) as,

$$\left.\begin{aligned} W_j(z_j) &= \frac{2}{1+\lambda_j} W_0(z_j) - \frac{1-\lambda_j}{1+\lambda_j} T_0^{\infty} && \left(\|z_j\| \leqslant R_j\right) \\ W(z_j) &= W_0(z_j) + \frac{1-\lambda_j}{1+\lambda_j}\left[\overline{W}_0\left(\frac{R_j^2}{z_j}\right) - T_0^{\infty}\right] && \left(R_j \leqslant \|z_j\|\right) \end{aligned}\right\} \tag{20}$$

The temperature potential $W_0(z_j)$ at infinity can be further expended into Taylor series such as,

$$W_0(z_j) = T_0^{\infty} + \sum_{n=0}^{\infty} T_{n+1}^{\infty} z_j^{n+1}, \quad \operatorname{Im}\left(T_0^{\infty}\right) = 0 \tag{21}$$

where T_0^{∞}, T_{n+1}^{∞} are unknown coefficients. If the temperature field at infinity is known, one can easily determine the unknown coefficients T_0^{∞}, T_{n+1}^{∞} by substituting $W_0(z_j)$ into eqn (1).

Obviously, the temperature complex potentials in eqn (20) constructed in this way satisfy the continuity conditions (15) and (16) and the boundary condition at infinity. When the potential $W_0(z_j)$ is determined by the temperature boundary condition at infinity, the temperature complex potentials in the matrix and the inhomogeneity are automatically determined by eqn (20).

2.3. Thermoelastic solution

Let the thermoelastic state of the matrix containing an inhomogeneity be subjected to a known temperature field. Since no singularities are assumed to reside inside or on the boundary of the inhomogeneity, $\phi_j(z_j)$ and $\psi_j(z_j)$ must be holomorphic in the inhomogeneity and $\phi(z_j)$ and $\psi(z_j)$ in an annulus region of the matrix bounded by certain concentric circles, as shown in Fig. 1. Assume these stress complex potentials can be expanded into Taylor and Laurent series in their respective regions (Isida, 1973; Meguid and Zhu, 1995), such that,

$$\left.\begin{aligned} \phi_j(z_j) &= \sum_{n=0}^{\infty} H_{n,j} z_j^{n+1} && \left(\|z_j\| \leqslant R_j\right) \\ \psi_j(z_j) &= \sum_{n=0}^{\infty} L_{n,j} z_j^{n+1} && \left(\|z_j\| \leqslant R_j\right) \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} \phi(z_j) &= -\Delta R_j^2 \bar{T}_1^{\infty} \ln(z_j) + \sum_{n=0}^{\infty}\left(M_{n,j} z^{n+1} + F_{n,j} z^{-(n+1)}\right) && \left(R_j \leqslant \|z_j\|\right) \\ \psi(z_j) &= -\Delta R_j^2 T_1^{\infty} \ln(z_j) + \sum_{n=0}^{\infty}\left(K_{n,j} z^{n+1} + D_{n,j} z^{-(n+1)}\right) && \left(R_j \leqslant \|z_j\|\right) \\ \Delta &\equiv \frac{A\left(1-\lambda_j\right)}{(1+\kappa)\left(1+\lambda_j\right)} \end{aligned}\right\} \tag{23}$$

where $H_{n,j}$, $L_{n,j}$, $M_{n,j}$, $F_{n,j}$, $K_{n,j}$, $D_{n,j}$ are unknown coefficients. The constant terms corresponding to rigid body displacements in eqns (22) and (23) have been ignored since they have no effect on the stresses. The first term in eqn (23) is introduced to ensure the single-value displacement condition.

By substituting eqns (22) and (23) into the stress and displacement continuity conditions (17) and (18) and comparing the coefficients of various powers of $e^{i\theta}$ at both sides, the following relations between the unknown coefficients in the complex potentials can be derived as

$$\left.\begin{aligned} H_{0,j} &= \frac{(1-\alpha_j)(1-\gamma_j\beta_j)}{1-\alpha_j\beta_j}M_{0,j} + \frac{(\alpha_j-\gamma_j)(1-\beta_j)}{1-\alpha_j\beta_j}\overline{M}_{0,j} + \Lambda_j T_0^\infty \\ H_{n,j} &= (1-\alpha_j)M_{n,j} + \frac{\Pi_j}{n+1}T_n^\infty \quad (n \geqslant 1) \\ L_{n,j} &= (1-\beta_j)K_{n,j} + (n+3)(\alpha_j-\beta_j)M_{n+2,j}R_j^2 - \Pi_j T_{n+2}^\infty R_j^2 \\ &\quad + \frac{\Omega_j(1-\lambda_j)}{(1+n)(1+\lambda_j)}T_{n+2}^\infty R_j^2 \quad (n \geqslant 0) \end{aligned}\right\} \tag{24}$$

and

$$\left.\begin{aligned} F_{n,j} &= -\beta_j\overline{K}_{n,j}R_j^{2n+2} - (n+3)\beta_j\overline{M}_{n+2,j}R_j^{2n+4} + \frac{\Omega_j(1-\lambda_j)}{(1+n)(1+\lambda_j)}\overline{T}_{n+2}^\infty R_j^{2n+4} \\ D_{0,j} &= -2\gamma_j \operatorname{Re}[M_{0,j}]R_j^2 + 2\Lambda_j T_0^\infty R_j^2 \\ D_{1,j} &= -\alpha_j\overline{M}_{1,j}R_j^4 + \left[\frac{\Pi_j}{2} + \frac{A(1-\lambda_j)}{(1+\kappa)(1+\lambda_j)}\right]\overline{T}_1^\infty R_j^4 \\ D_{n,j} &= -(n-1)\beta_j\overline{K}_{n-2,j}R_j^{2n} - [(n^2-1)\beta_j + \alpha_j]\overline{M}_{n,j}R_j^{2n+2} \\ &\quad + \frac{\Pi_j}{1+n}\overline{T}_n^\infty R_j^{2n+2} + \Omega_j\frac{1-\lambda_j}{1+\lambda_j}\overline{T}_n^\infty R_j^{2n+2} \quad (n \geqslant 2) \end{aligned}\right\} \tag{25}$$

where

$$\left.\begin{aligned} &\alpha_j = \frac{\kappa_j - \Gamma_j\kappa}{\Gamma_j + \kappa_j} \quad \beta_j = \frac{1-\Gamma_j}{1+\Gamma_j\kappa} \quad \gamma_j = \frac{\alpha_j - \beta_j}{(1-\beta_j) - \beta_j(1-\alpha_j)} \\ &\Omega_j = \frac{\Gamma_j A}{1+\Gamma_j\kappa} \quad \Lambda_j = \frac{\Gamma_j A - A_j}{2\Gamma_j + \kappa_j - 1} \quad \Pi_j = \frac{A}{\Gamma_j + \kappa_j}\left(\Gamma_j - \frac{2A_j}{(1+\lambda_j)A}\right) \end{aligned}\right\} \tag{26}$$

Eqns (22)–(25) constitute the general expressions of the stress complex potentials which satisfy the continuity conditions across the interface between the inhomogeneity and the matrix for a single circular inhomogeneity. The thermoelastic problem is thus reduced to solve the two independent unknown coefficients $M_{n,j}$ and $K_{n,j}$ with specific geometry and external force conditions of the surrounding matrix. Once they are known, the stresses and displacements throughout the whole field are automatically determined by eqns (5)–(7) and (22)–(25).

In order to verify the present solution, let us consider a single circular inhomogeneity in an infinite matrix subjected only to a linear temperature change at infinity. An inhomogeneity of radius R_1 is assumed to be centered at the origin O_1 of the local coordinate systems (x_1, y_1) and (r_1, θ_1). The temperature state at infinity is given in the local coordinate system as

$$T^{\infty} = T_0 - \frac{q}{k} r_1 \cos(\theta_1 - \omega) \tag{27}$$

where T_0, q and ω represent the uniform temperature change, constant heat flux and the angle which the direction of heat flux makes with the positive x_1-axis.

By applying the solution derived above, the closed form solution for temperature and thermal stresses in the matrix induced by the presence of the inhomogeneity is obtained as

$$T = T_0 - \frac{q}{k}\left[r_1 \cos(\theta_1 - \omega) + \frac{1-\lambda_1}{1+\lambda_1}\frac{R_1^2}{r_1}\cos(\theta_1 - \omega)\right]$$

$$\sigma_r = \Lambda_1 T_0 \frac{R_1^2}{r_1^2} - \frac{A}{1+\kappa}\frac{qR_1}{\kappa}\left[\frac{1+\kappa}{A}\frac{\Pi_1}{2}\frac{R_1^3}{r_1^3} - \frac{1-\lambda_1}{1+\lambda_1}\left(1 - \frac{R_1^2}{r_1^2}\right)\frac{R_1}{r_1}\right]\cos(\theta_1 - \omega)$$

$$\sigma_\theta = -\Lambda_1 T_0 \frac{R_1^2}{r_1^2} + \frac{A}{1+\kappa}\frac{qR_1}{\kappa}\left[\frac{1+k}{A}\frac{\Pi_1}{2}\frac{R_1^3}{r_1^3} + \frac{1-\lambda_1}{1+\lambda_1}\left(1 + \frac{R_1^2}{r_1^2}\right)\frac{R_1}{r_1}\right]\cos(\theta_1 - \omega)$$

$$\tau_{r\theta} = -\frac{A}{1+\kappa}\frac{qR_1}{k}\left[\frac{1+\kappa}{A}\frac{\Pi_1}{2}\frac{R_1^3}{r_1^3} + \frac{1-\lambda_1}{1+\lambda_1}\left(1 - \frac{R_1^2}{r_1^2}\right)\frac{R_1}{r_1}\right]\sin(\theta_1 - \omega)$$

The stress solution reduces to the solution of Kattis and Meguid (1995) under the condition of plane stress and $\omega = \pi/2$. Under more specific assumption of a rigid inclusion ($\Lambda_1 = 2\mu(a - a_1)$, $\Pi_1 = 4\mu[a - 2a_1(1 + \lambda_1)]$) or a traction-free and insulated hole ($\Lambda_1 = \Pi_1 = 0$), the above stress solution reduces to the exiting solutions of Lee and Choi (1989) or Florence and Goodier (1960), respectively.

3. Solution of multiple interacting inhomogeneities

3.1. Heat conduction solution of multiple inhomogeneities

Now let us extend our attention to the problem of multiple interacting inhomogeneities. Consider an infinite matrix containing N distinct circular inhomogeneities subjected to an arbitrary thermal loading at infinity as shown in Fig. 1. Let (x_j, y_j) and (r_j, θ_j) denote the rectangular Cartesian and polar coordinate systems with their origin O_j at the center of the j-th inhomogeneity whose radius is R_j $(j = 1, 2, \dots, N)$. The distance between the j-th and k-th inhomogeneities is denoted by d_{jk} and the inclination angle measured from x_j-axis to the line O_jO_k by θ_{jk}. Define the dimensionless coordinates, complex variables and parameters by

$$\zeta_j = \frac{z_j}{d}, \quad l_{jk} = \frac{d_{jk}}{d}, \quad \frac{R_j}{d} = \varepsilon S_j, \quad S_j = \frac{R_j}{R}, \quad \varepsilon = \frac{R}{d} \tag{28}$$

where d and R are arbitrary reference length and radius.

From eqn (20), the temperature complex potential in the matrix is defined as the following sum of $(N+1)$ functions:

$$W(\zeta_j) = W_0(\zeta_j) + \sum_{k=1}^{N} \frac{1-\lambda_k}{1+\lambda_k}\left[\overline{W}_k\left(\frac{S_k^2\varepsilon^2}{\zeta_k}\right) - T_{0,k}\right] \quad \left(\varepsilon S_j \leqslant \|\zeta_j\|\right) \tag{29}$$

with

$$W_0 = T_0^{\infty} + \sum_{n=0}^{\infty} T_{n+1}^{\infty}\zeta_j^{n+1}, \quad \mathrm{Im}\left(T_0^{\infty}\right) = 0 \tag{30}$$

where $W_0(\zeta_j)$ is a holomorphic temperature complex potential in j-th inhomogeneity coordinates, corresponding to the temperature state at infinity in the absence of the inhomogeneity.

Since no heat source exists inside the inhomogeneity, the potential $W_k(\zeta_k)$ representing the presence of k-th inhomogeneity is holomorphic and can be expressed in terms of k-th inhomogeneity coordinates in Taylor series form as

$$W_k(\zeta_k) = T_{0,k} + \sum_{n=0}^{\infty} T_{n+1,k}\zeta_k^{n+1}, \quad \mathrm{Im}\left(T_{0,k}\right) = 0 \quad \left(\|\zeta_k\| \leqslant \varepsilon S_k\right) \tag{31}$$

where $T_{n,k}$ are unknown coefficients to be determined.

The temperature potential $W(\zeta_j)$ constructed in this manner satisfies automatically the boundary conditions at infinity and only the continuity condition along the interface of inhomogeneity needs to be checked. To check the local continuity condition, it is convenient to express the temperature potential $W(\zeta_j)$ in terms of j-th inhomogeneity local coordinates. The transformation relationship between the j-th and the k-th inhomogeneity coordinates is

$$\zeta_k = \zeta_j - l_{jk}\,\mathrm{e}^{i\theta_{jk}} \tag{32}$$

Substituting eqn (32) into (29) and expanding it into Taylor series form with respect to ζ_j in an annulus region of $\varepsilon S_j \leqslant \|\zeta_j\| \leqslant \varepsilon$, the temperature potential $W(\zeta_j)$ in eqn (29) is reduced to

$$W(\zeta_j) = T_0^{\infty} + \sum_{n=0}^{\infty} T_{n+1,j}\zeta_j^{n+1} + \frac{1-\lambda_j}{1+\lambda_j}\sum_{n=0}^{\infty} \overline{T}_{n+1,j}\frac{S_j^{2n+2}}{\zeta_j^{n+1}}\varepsilon^{2n+2} \tag{33}$$

where

$$\left.\begin{aligned} T_{n,j} &= T_n^{\infty} + \sum_{p=0}^{\infty}\sum_{k\neq j}^{N} \frac{1-\lambda_k}{1+\lambda_k} a_{n-1,\,j}^{p,k}\,\overline{T}_{p+1,\,k} S_k^{2p+2}\varepsilon^{2p+2} \quad (n = 0, 1, 2, \dots) \\ a_{n,j}^{p,k} &= \frac{(-1)^{p+1}}{l_{jk}^{n+p+2}}\binom{n+p+1}{p}\mathrm{e}^{-i(n+p+2)\theta_{jk}} \end{aligned}\right\} \tag{34}$$

In order to solve the unknown coefficients $T_{n,j}$, assume they can be expressed as the power series of ε^{2q},

$$T_{n,j} = \sum_{q=0}^{\infty} T_{n,j}^{(2q)}\varepsilon^{2q} \tag{35}$$

where $T_{n,j}^{(2q)}$ are another set of unknown coefficients to be determined. By substituting eqn (35) into (34) and comparing the coefficients of various powers of ε^{2q} at both sides, the recurrence formulae of $T_{n,j}^{(2q)}$ are obtained as:

$$\left.\begin{aligned} & T_{n,j}^{(0)} = T_n^{\infty} \\ & T_{n,j}^{(2q)} = \sum_{k \neq j}^{N} \sum_{p=0}^{q-1} \frac{1-\lambda_k}{1+\lambda_k} a_{n-1,j}^{p,k} S_k^{2p+2} \overline{T}_{p+1,k}^{(2q-2p-2)} \quad (n \geqslant 0,\ q \geqslant 1) \end{aligned}\right\} \tag{36}$$

Eqn (36) constitutes the necessary consistency equations for determining the unknown coefficients $T_{n,j}\ (n = 0, 1, 2, \ldots; j = 1, 2, \ldots, N)$ successively. The temperature potential inside the j-th inhomogeneity can then be easily determined by eqn (20).

3.2. Thermoelastic stress solution of multiple inhomogeneities

Assume the Airy's stress function in the matrix is expressed as the following sum of $N + 1$ functions

$$U = U_0 + \sum_{k=1}^{N} U_k \tag{37}$$

where U_0 represents the stress state at infinity in the homogeneous matrix and U_k the presence of k-th inhomogeneity. The stress function U_k is defined in the k-th inhomogeneity coordinates, such as,

$$\left.\begin{aligned} & U_k = \mathrm{Re}\left[\bar{\zeta}_k \underline{\phi}_k(\zeta_k) + \underline{\chi}_k(\zeta_k)\right] \quad \text{with} \quad \underline{\chi}_k'(\zeta_k) = \underline{\psi}_k(\zeta_k) \\ & \underline{\phi}_k(\zeta_k) = -\frac{A}{1+\kappa}\frac{1-\lambda_k}{1+\lambda_k} S_k^2 \varepsilon^2 \overline{T}_{1,k} \ln(\zeta_k) + \sum_{n=0}^{\infty} F_{n,k} \zeta_k^{-(n+1)} \quad (\varepsilon S_k \leqslant \|\zeta_k\|) \\ & \underline{\psi}_k(\zeta_k) = -\frac{A}{1+\kappa}\frac{1-\lambda_k}{1+\lambda_k} S_k^2 \varepsilon^2 T_{1,k} \ln(\zeta_k) + \sum_{n=0}^{\infty} D_{n,k} \zeta_k^{-(n+1)} \quad (\varepsilon S_k \leqslant \|\zeta_k\|) \end{aligned}\right\} \tag{38}$$

The first term in eqn (38) is introduced to ensure the single-value displacement condition. The Airy's stress function U in eqn (37) constructed in this way satisfies automatically the boundary conditions at infinity and only the local continuity condition across the interface between the inhomeneity and the matrix needs to be checked. If we concentrate our attention on thermal effect only and assume a free stress state at infinity, the stress function U_0 in eqn (37) becomes zero.

Substituting coordinate transformation (32) into (38), expanding the stress functions into Laurent series with respect to variable ζ_j in an annulus region of $\varepsilon S_j \leqslant \|\zeta_j\| \leqslant \varepsilon$, and omitting the constant terms which have no contribution to stresses, the stress functions in eqn (38) are reduced to the same form of the stress functions for one circular inhomogeneity in eqn (23)

$$U = \mathrm{Re}\left[\bar{\zeta}_j \phi(\zeta_j) + \chi(\zeta_j)\right] \quad \text{with} \quad \chi'(\zeta_k) = \psi(\zeta_k)$$

$$\left.\begin{aligned}\phi(\zeta_j) &= -\frac{A}{1+\kappa}\frac{1-\lambda_j}{1+\lambda_j}S_j^2\varepsilon^2\overline{T}_{1,j}\ln(\zeta_j) + \sum_{n=0}^{\infty}\left(M_{n,j}\zeta_j^{n+1} + F_{n,k}\zeta_j^{-(n+1)}\right) \quad (\varepsilon S_j \leqslant \|\zeta_j\| \leqslant \varepsilon) \\ \psi(\zeta_j) &= -\frac{A}{1+\kappa}\frac{1-\lambda_j}{1+\lambda_j}S_j^2\varepsilon^2 T_{1,j}\ln(\zeta_j) + \sum_{n=0}^{\infty}\left(K_{n,j}\zeta^{n+1} + D_{n,j}\zeta_j^{-(n+1)}\right) \quad (\varepsilon S_j \leqslant \|\zeta_j\| \leqslant \varepsilon)\end{aligned}\right\} \tag{39}$$

where

$$\left.\begin{aligned}M_{n,j} &= \sum_{k\neq j}^{N} c_{n,j}^{k}\overline{T}_{1,k}S_k^2\varepsilon^2 + \sum_{p=0}^{\infty}\sum_{k\neq j}^{N} a_{n,j}^{p,k}F_{p,k} \\ K_{n,j} &= \sum_{k\neq j}^{N}\left[T_{1,k} - (n+1)\,\mathrm{e}^{-i2\psi_{jk}}\overline{T}_{1,k}\right]c_{n,j}^{k}S_k^2\varepsilon^2 + \sum_{p=0}^{\infty}\sum_{k\neq j}^{N}\left(a_{n,j}^{p,k}D_{p,k} + b_{n,j}^{p,k}F_{p,k}\right) \\ b_{n,j}^{p,k} &= \frac{(-1)^p}{l_{jk}^{n+p+2}}\binom{n+p+2}{p}(n+2)\,\mathrm{e}^{-i(n+p+4)\theta_{jk}} \\ c_{n,j}^{k} &= \frac{A}{1+\kappa}\frac{1-\lambda_k}{1+\lambda_k}\frac{\mathrm{e}^{-i(n+1)\theta_{jk}}}{(n+1)l_{jk}^{n+1}}\end{aligned}\right\} \tag{40}$$

Analog to the solution for the single inhomogeneity, the stress functions inside the j-th inhomogeneity can be expressed as

$$\left.\begin{aligned}\phi_j(\zeta_j) &= \sum_{n=0}^{\infty} H_{n,j}\zeta_j^{n+1} \quad (\|\zeta_j\| \leqslant \varepsilon S_j) \\ \psi_j(\zeta_j) &= \sum_{n=0}^{\infty} L_{n,j}\zeta_j^{n+1} \quad (\|\zeta_j\| \leqslant \varepsilon S_j)\end{aligned}\right\} \tag{41}$$

Substituting eqns (39)–(41) into the local stress and displacement continuity conditions (17) and (18) along the inhomogeneity/matrix interface and comparing the coefficients of various powers of $\mathrm{e}^{i\theta}$ at both sides, we obtain the same relationship among the coefficients $H_{n,j}$, $L_{n,j}$, $F_{n,j}$, $D_{n,j}$, $M_{n,j}$, and $K_{n,j}$ as shown in eqns (24) and (25).

In order to solve the unknown coefficients $M_{n,j}$, $K_{n,j}$, $F_{n,j}$ and $D_{n,j}$, assume they can be expressed as the power series of ε^{2q}, such that

$$\left.\begin{aligned}M_{n,j} &= \sum_{q=0}^{\infty} M_{n,j}^{(2q)}\varepsilon^{2q}, \quad K_{n,j} = \sum_{q=0}^{\infty} K_{n,j}^{(2q)}\varepsilon^{2q} \\ F_{n,j} &= \sum_{q=n+1}^{\infty} F_{n,j}^{(2q)}\varepsilon^{2q}, \quad D_{n,j} = \sum_{q=n}^{\infty} D_{n,j}^{(2q)}\varepsilon^{2q}\end{aligned}\right\} \tag{42}$$

where $M_{n,j}^{(2q)}$, $K_{n,j}^{(2q)}$, $F_{n,j}^{(2q)}$, $D_{n,j}^{(2q)}$ are new sets of unknown coefficients to be determined. Substituting eqn (42) into (25) and (40) and comparing the coefficients of various powers of ε^{2q} at both sides, we obtain the following recurrence formulae for these new coefficients:

$$\left.\begin{aligned}
&M_{n,j}^{(0)} = K_{n,j}^{(0)} = 0 \quad (n \geqslant 0)\\
&F_{n,j}^{(2n+2)} = 0 \quad (n \geqslant 0)\\
&D_{0,j}^{(0)} = D_{1,j}^{(2)} = D_{n,j}^{(2n)} = 0 \quad (n \geqslant 2)\\
&M_{n,j}^{(2)} = \sum_{k \neq j}^{N} \left(a_{n,j}^{0,k} F_{0,k}^{2} + c_{n,j}^{k} \overline{T}_{1,k} S_{k}^{2} \right) \quad (n \geqslant 0)\\
&K_{n,j}^{(2)} = \sum_{k \neq j}^{N} \left(a_{n,j}^{0,k} D_{0,k}^{2} + b_{n,j}^{0,k} F_{0,k}^{(2)} \right) + \sum_{k \neq j}^{N} \left[T_{1,k} - (n+1)\, \mathrm{e}^{-i2\theta_{jk}} \overline{T}_{1,k} \right] c_{n,j}^{k} S_{k}^{2} \quad (n \geqslant 0)\\
&F_{n,j}^{(2n+2q)} = -\beta_j \overline{K}_{n,j}^{(2q-2)} S_j^{2n+2} - (n+3)\beta_j \overline{M}_{n+2,j}^{(2q-4)} S_j^{2n+4}\\
&\qquad + \frac{\Omega_j}{1+n} \frac{1-\lambda_j}{1+\lambda_j} \overline{T}_{n+2,j} S_j^{2n+4} \delta_{2,q} \quad (n \geqslant 0,\ q \geqslant 2)\\
&D_{0,j}^{(2q)} = -2\gamma_j \mathrm{Re}\left[M_{0,j}^{(2q-2)} \right] S_j^2 + 2\Lambda_j T_{0,j} S_j^2 \delta_{1,q} \quad (q \geqslant 1)\\
&D_{1,j}^{(2q)} = -\alpha_j \overline{M}_{1,j}^{(2q-4)} S_j^4 + \left(\frac{\Pi_j}{2} + \frac{A}{1+\kappa} \frac{1-\lambda_j}{1+\lambda_j} \right) \overline{T}_{1,j} S_j^4 \delta_{2,\,q} \quad (q \geqslant 2)\\
&D_{n,j}^{(2n+2q-2)} = -(n-1)\beta_j \overline{K}_{n-2,j}^{(2q-2)} S_j^{2n} - \left[(n^2-1)\beta_j + \alpha_j \right] \overline{M}_{n,j}^{(2q-4)} S_j^{2n+2}\\
&\qquad + \left(\frac{\Pi_j}{1+n} + \Omega_j \frac{1-\lambda_j}{1+\lambda_j} \right) \overline{T}_{n,j} S_j^{2n+2} \delta_{2,q} \quad (n \geqslant 2,\ q \geqslant 2)\\
&M_{n,j}^{(2q)} = \sum_{k \neq j}^{N} \sum_{p=0}^{q-1} a_{n,j}^{p,k} F_{p,k}^{(2q)} \quad (n \geqslant 0,\ q \geqslant 2)\\
&K_{n,j}^{(2q)} = \sum_{k \neq j}^{N} \sum_{p=0}^{q} a_{n,j}^{p,k} D_{p,k}^{(2q)} + \sum_{k \neq j}^{N} \sum_{p=0}^{q-1} b_{n,j}^{p,k} F_{p,k}^{(2q)} \quad (n \geqslant 2,\ q \geqslant 2)
\end{aligned}\right\} \tag{43}$$

where $\delta_{i,j}$ is the Kronecker delta.

Eqns (40), (42) and (43) constitute the necessary consistency equations for determining the unknown coefficients $M_{n,j}$, $K_{n,j}$, $F_{n,j}$, and $D_{n,j}$ $(n = 1, 2, \ldots;\ j = 0, 1, 2, \ldots, N)$. The coefficients $H_{n,j}$ and $L_{n,j}$ for the j-th inhomogeneity are then determined with eqn (24). All the unknown coefficients in the complex potentials can be determined successively as accurate as required.

4. Applications and discussions

The solution developed above was applied to some typical cases in this Section to show its capability in solving problems of multiple interacting inhomogeneities. Since the closed form solution no longer exists for the problems involving two or more inhomogeneities, an alternative numerical procedure is adopted to obtain the solution. The numerical solution yields the coefficients of the complex stress potentials $T_{n,j}$, $M_{n,j}$, $K_{n,j}$, $F_{n,j}$, $D_{n,j}$, $H_{n,j}$, $L_{n,j}$. Then, the stresses and the displacements in the matrix and the inhomogeneity can be determined by utilizing the complex stress potentials. One important thermal loading, uniform temperature change, is considered here under the plane strain conditions. It has also been assumed that the value of Poisson's ratio equals to 0.3 for both the matrix and the inhomogeneity. Finally, the thermal stress is normalized by $\sigma_0 = 2\mu(a_j - a)T$.

4.1. Two identical inhomogeneities

The present solution is first verified by two inhomogeneity problems. The two identical inhomogeneities are placed on the x-axis with a center-to-center distance between inhomogeneities of $10R/3$, where R is the radius of the inhomogeneity. The results of thermal stress σ_θ in the matrix along the inhomogeneity interface are plotted in Fig. 2 for discrete values of the elastic moduli. It is shown that the presence of the inhomogeneities causes the stress concentration along the interface of the inhomogeneity. The stiffer the inhomogeneity, the higher the stress concentration. Also shown in the figure is the interaction between the inhomogeneities, which affects the stress concentration along the interface. This is evidenced by the different stress values at $\theta = 0°$ and 180°, which should be equal if there were only one inhomogeneity. The present solution agrees with the existing result of Kouris and Tsuchida (1991) exactly.

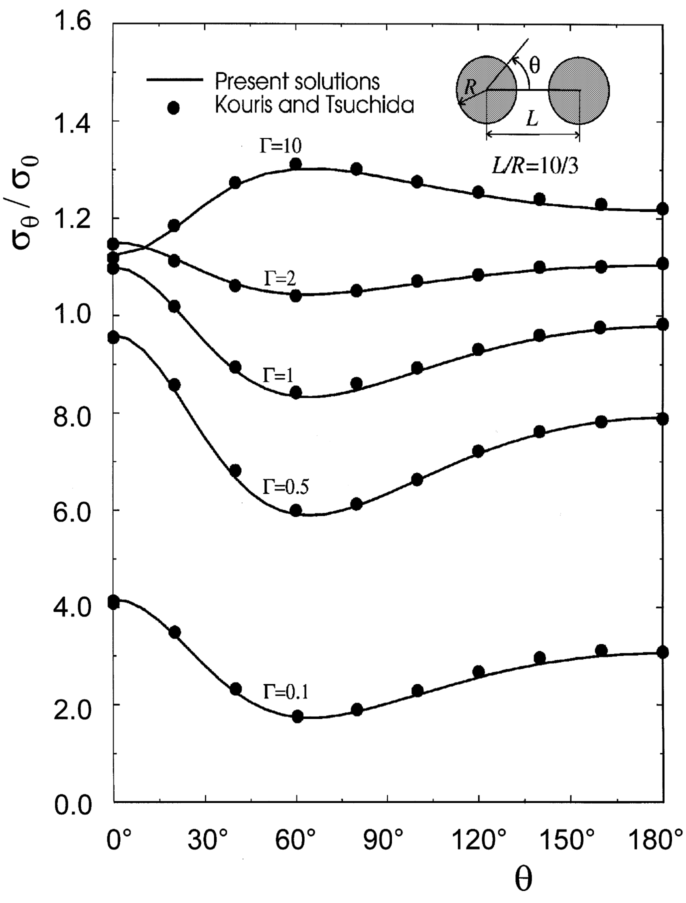

Fig. 2. Distribution of hoop stress along the left inhomogeneity interface in the matrix for two identical inhomogeneities.

4.2. Three identical inhomogeneities

The second example is the case involving three identical inhomogeneities embedded in an infinite matrix. The centers of the inhomogeneities are at the vertices of an equilateral triangle with sides equal to $3R$, where R is the radius of the inhomogeneity. Due to the symmetry of the geometry and loading condition, the results of the thermal stresses are presented for different discrete values of elastic moduli along one half of the interface of the right inhomogeneity. Fig. 3 shows the variation of the hoop stress σ_θ at both sides of matrix–inhomogeneity interface. The solid lines represent the interfacial hoop stress in the matrix, while the dashed lines show it in the inhomogeneity. Compared with the results of the two inhomogeneities under the same thermal loading condition, more interacting effects are observed among the inhomogeneities leading to higher concentration of the stress. The stress concentration is proportional to the elastic moduli of the inhomogeneity, as shown in the previous example. The mismatch in the elastic and thermal properties of the two materials introduces tensile/compressive stresses along the interface. This is evidenced by the presence of positive and negative hoop stresses at the two sides of the interface. Interestingly, it is noticed that the stress concentration for hoop stress at the matrix side is insensitive to the neighboring inhomogeneities when the elastic mismatch $\Gamma = 3$.

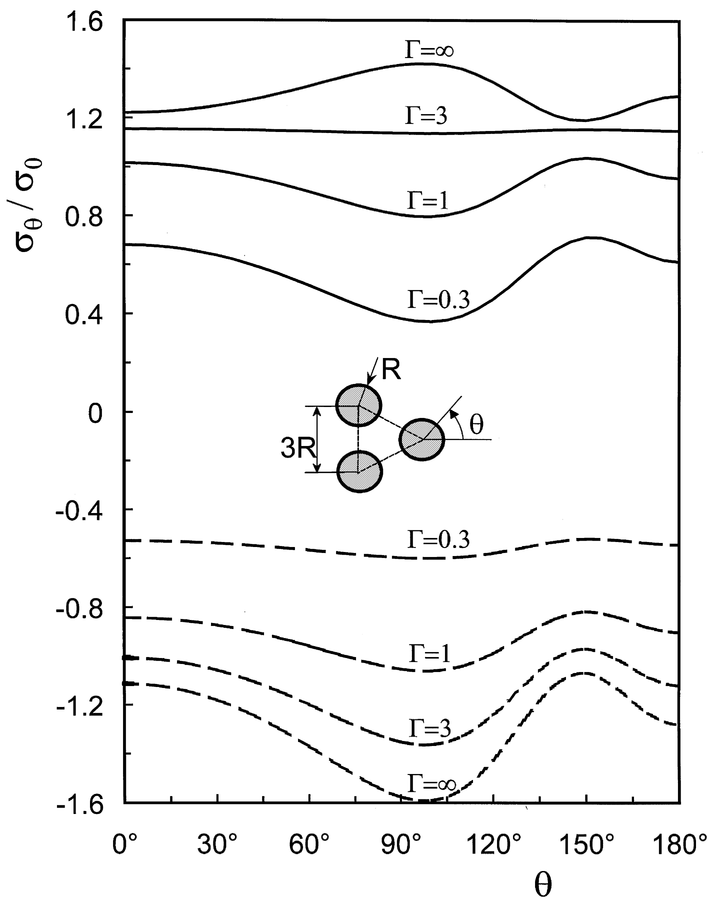

Fig. 3. Distribution of hoop stress along the right inhomogeneity interface for three identical inhomogeneities.

Correspondingly, the stress concentration for hoop stress at the inhomogeneity side is less sensitive to the neighboring inhomogeneities when the elastic mismatch $\Gamma = 0.3$.

The radial and shear stresses are shown to be continuous in Fig. 4. The shear stresses vary around zero, while the radial stresses are all negative. If the coefficient of thermal expansion of the inhomogeneity is greater than that of the matrix, the nominal stress σ_0 is positive. Thus, the negative normalized radial stress suggests a compressive stress, because the inhomogeneity expands faster than the surrounding matrix. The mismatch of the elastic and thermal properties also introduces a stress concentration. Similar to the hoop stress, the stress concentration for the radial and shear stresses is also proportional to the elastic moduli of the inhomogeneity.

4.3. Five identical inhomogeneities

The next example involves the case of five identical inhomogeneities embedded in an infinite matrix. Four inhomogeneities are located at the vertices of a square and one inhomogeneity at the center of the square. The center-to-center distances between the central inhomogeneity and the surrounding ones are equal to three times of the radius of the inhomogeneity, $3R$. Fig. 5 shows the variation of the hoop stress σ_θ at both sides of the interface. The solid lines represent the interfacial hoop stresses in the matrix, while the dashed lines show it in the inhomogeneity. The interfacial radial and shear stresses are shown in Fig. 6 in which the shear stresses vary around the zero and the radial stresses are all negative. Very strong interacting effects among the inhomogeneities are evidenced by the presence of more variations in the stress concentration values. Again, the stress concentration for the hoop stress at the matrix side is insensitive to the neighboring inhomogeneities when the elastic mismatch $\Gamma = 3$.

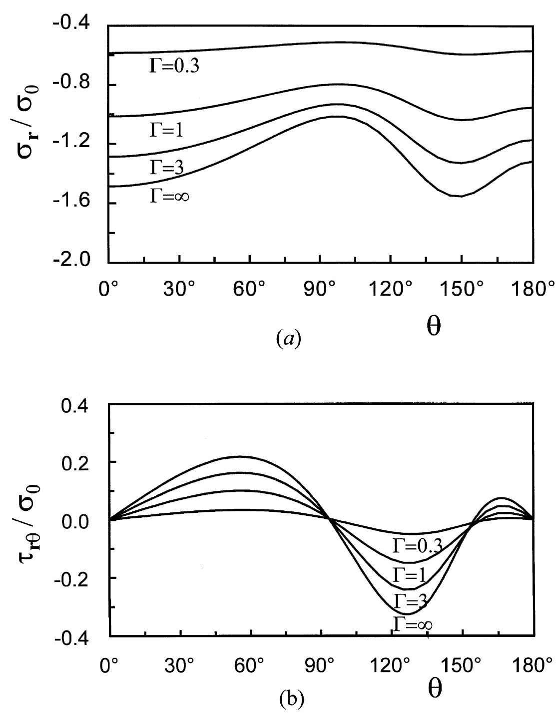

Fig. 4. Distribution of stresses along the right inhomogeneity interface for three identical inhomogeneities: (a) radial stress distribution; and (b) shear stress distribution.

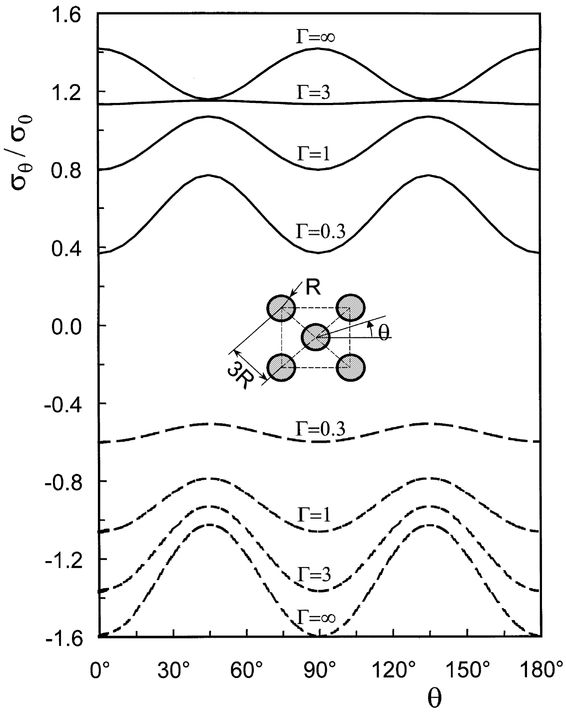

Fig. 5. Distribution of hoop stress along the central inhomogeneity interface for five identical inhomogeneities.

4.4. Five different inhomogeneities

In the final case, we examined the thermal stresses resulting from five different inhomogeneities in an infinite matrix. The spatial configuration of the inhomogeneities is the same as the previous example of five identical inhomogeneities. The size of the four inhomogeneities centering at the vertices of the square are the same and their radii are R. The center-to-center distances between the central inhomogeneity and the surrounding ones are equal to $3R$. The size of the central inhomogeneity varies from $0.5R$ to $1.8R$. The material properties of each inhomogeneity, which is normalized by the material property of the matrix, are listed as follows

Inhomogeneity	Shear modulus Γ	Thermal expansion a
Center	4	0.16
Right	20	0.08
Left	2	0.64

The results of the thermal stresses along the interface of the central inhomogeneity with varying radii are calculated using the newly developed solution. The hoop stresses are shown in Fig. 7. The solid lines represent the interfacial hoop stress in the matrix, while the dashed lines show it in the inhomogeneity. The interfacial radial and shear stresses are depicted in Fig. 8. When the central inhomogeneity is quite small ($r/R = 0.5$), interacting effects among the different inhomogeneities are very weak. These interacting effects increase significantly as the central inhomogeneity becomes larger ($r/R = 1.8$). The

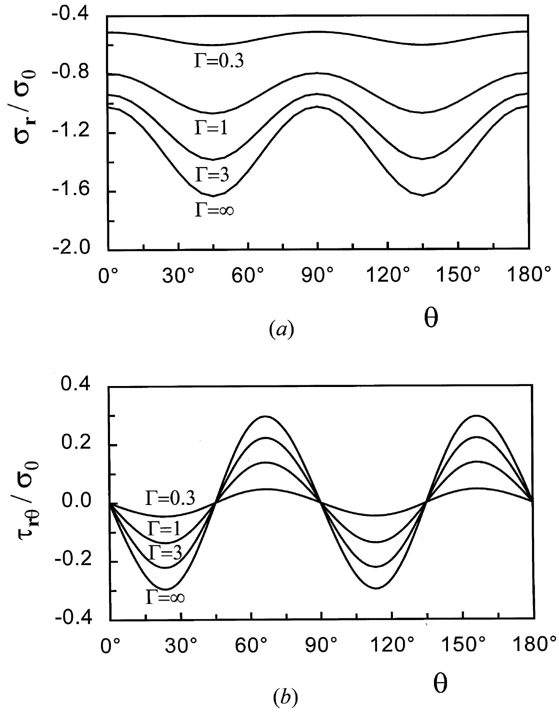

Fig. 6. Distribution of stresses along the central inhomogeneity interface for five identical inhomogeneities: (a) radial stress distribution; and (b) shear stress distribution.

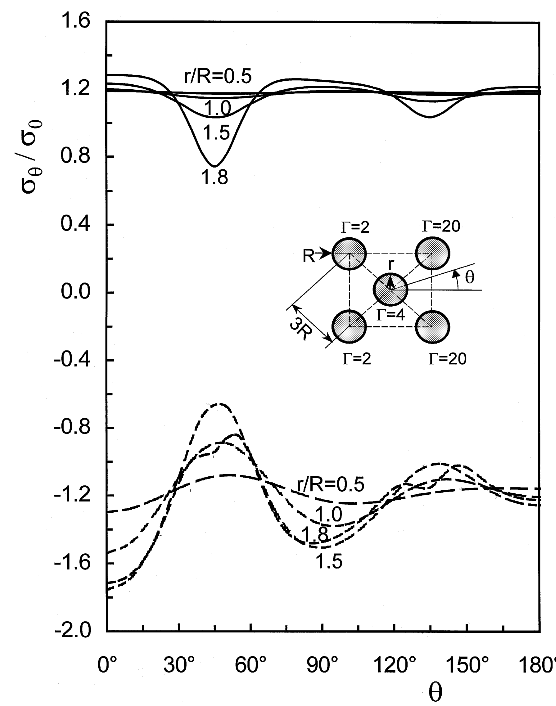

Fig. 7. Distribution of hoop stress along the central inhomogeneity interface for five different inhomogeneities.

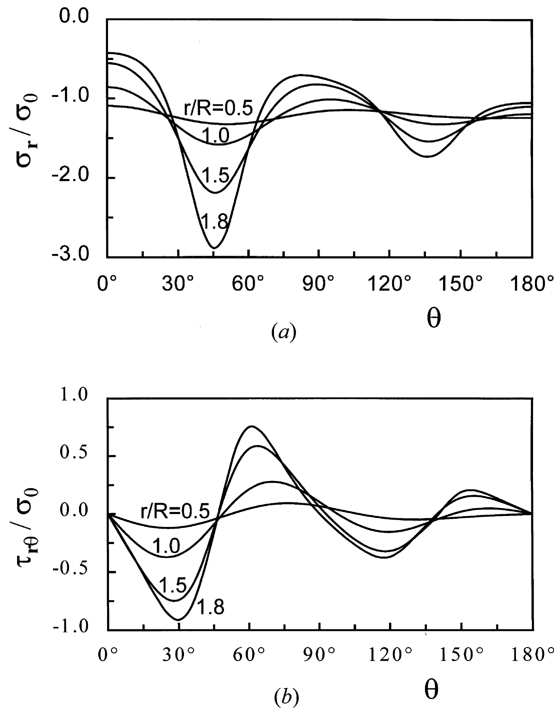

Fig. 8. Distribution of stresses along the central inhomogeneity interface for five different inhomogeneities: (a) radial stress distribution; and (b) shear stress distribution.

stress concentrations vary more significantly in the region of $0^\circ \leqslant \theta \leqslant 90^\circ$, where the central inhomogeneity faces the harder inhomogeneity ($\Gamma = 20$), than in the region of $90^\circ \leqslant \theta \leqslant 180^\circ$, where the central inhomogeneity faces the softer inhomogeneity ($\Gamma = 2$).

5. Conclusion

The present work provides a general solution to the problem of thermoelasticity of the multiple interacting inhomogeneities embedded in an elastic matrix. The solution, which based upon the complex potentials of Muskhelishvili and Laurent series expansion method for both heat conduction and thermoelasticity problems, gives a general expression of the proposed complex potentials in the circular inhomogeneities and the surrounding matrix under arbitrary thermal loading conditions. The main feature of the approach is the repeated use of the solution for the single inhomogeneity problem by an appropriate superposition. It reduces the multiple inhomogeneity problem to a system of linear algebraic equations which can be solved with perturbation technique. The results of the work should be of interest to those working in the fields of micromechanics and damage theory of composite materials. Furthermore, if the stress function U_0 in eqn (37) is determined by the stress state at infinity, the present solution can also be applied to the case of combined thermal loading and external forces. A number of examples are presented in the paper. The validity of the present study is verified by the excellent coincidence of the present and existing solutions for the problems containing one or two inhomogeneities. The capability of the present method in solving multiple interacting inhomogeneity problems is demonstrated by problems involving three and five inhomogeneities.

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